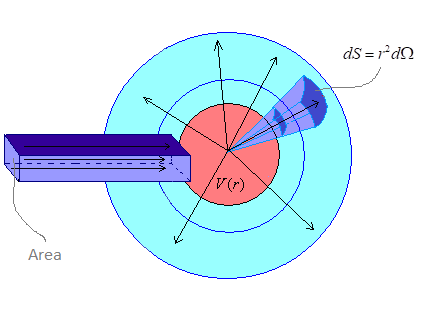
**Scattering**

Going to look at scattering. Our classical discussion below should pair well with the quantum mechanical one we’ll look at when we get to quantum mechanics.

**Scattering Cross Sections**

Say we send a beam of particles, with cross-section area, Area, into a target:



Some will scatter into some solid angle (Ω,Ω+dΩ), and some (perhaps, if Area is large enough), will not scatter at all (or will, in other words, simply *forward* scatter). The cross-section area of the beam which scatters, σ, is called the scattering cross-section area of the beam (parenthetically, we could say the transmission coefficient is T = 1 – R and R = σ/Area). For tiny beams, σ is just equal to Area. But as Area grows, eventually we’ll get to the point (perhaps) that some particles just miss the target entirely/are not scattered/are forward scattered. This target-dependent maximum scattered area is called the total scattering cross-section σT. And so for any Area > σT, we will still have σ = σT. The total scattering cross-section area can be thought of as the cross-section area within which the target’s force is active. For a contact force, the total scattering cross-section will be σT = πR2, where R is the radius of the target ball, at least classically. If the force is long-ranged, but exponentially damped, like Fexp(-r/μ), then the total scattering cross-section is still something like σT ~ πμ2 I think. But if the force goes as a power law, like F(μ/r)n, then the total scattering cross-section will truly be infinite. But while σT may be infinite sometimes, we should observe that σ itself will never be. We must always have σ ≤ Area.

We can examine the probability distribution of the *scattered* beam, P(Ω≠0,σ). We don’t concern ourselves with the *unscattered* (i.e., transmitted) particles, if any, as their behavior is known, and in any event, it is impossible to include them in the probability distribution P(Ω,σ) because as Area grows beyond σ­T, all particles added to the beam will simply *forward* scatter, and so we’ll just be changing the distribution at the single Ω = 0 point…but a probability distribution function isn’t responsive to change at a *single* point.

So suppose we have a given incident scattering cross section, σ, with incoming current Iinc., and we want to know what fraction of this current (i.e., the incident current that gets scattered) will scatter into angle Ω within width dΩ. This would be what?



where we define the ‘differential’ scattering cross-section:



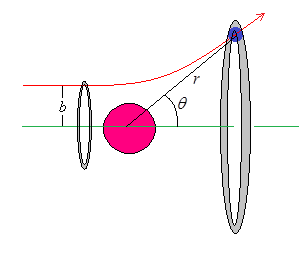
and in this context, interpret the dσ as the amount of scattering cross-section area of the beam, σ, that get’s scattered into the solid angle dΩ = sinθdθdφ. And so,



So all we have to do is divide dσ/dΩ by σ to get the probability distribution. We can accommodate having both the beam and the target moving, like is done with particle colliders where we send two beams of particles towards each other. The particles would be coming at each other with initial radial separation ri ~ ∞, and angular separation φi ~ π. And after the collision would emerge with radial separation r(t) → ∞, and angular separation φ(t) → φf. The case where both particles are moving can be handled just as the stationary target I think. We would just analyze the cross-section with the reduced mass in the moving case.

**Formula for differential scattering cross-section**

This formula is more convenient for the QM section, and less so here as we don’t typically form expressions for the current. Let’s look at it a different way. So typically a particle will be characterized by an impact parameter b, and will be ejected at an angle θ(b), dependent upon its kinetic energy usually, but not always:



So how can we get P(Ω)dΩ from b(θ)? Let’s presume the beam has a scattering cross-section area σ = π bmax2. Then,



and I suppose we can then write, dividing by dΩ = 2πsinθdθ (for azimuthally symmetric scattering)



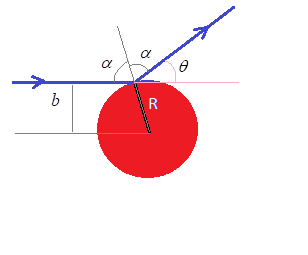
and the differential scattering cross section would be:



Note that this/these formula(s) will only be valid for b up to bmax, and so there will be an effective small angle cut-off for long range forces.

**Example: Hard Sphere Scattering**

Let’s do a simple example first – hard sphere scattering. Suppose we have an incoming beam with cross section area Area (in diagram only one particle within entire beam cross section is illustrated):



So particle at impact parameter b will be scattered into angle,



We already figured that out in a previous file. But so,



And therefore,



So the scattering cross-section is independent of angle? If we integrate over all dΩ. Then we should get the scattering cross-section area of the beam, σ:



And the probability distribution would be:



What if the beam cross-section area were larger than πR2. How much forward scattering would we get? So we’d have:



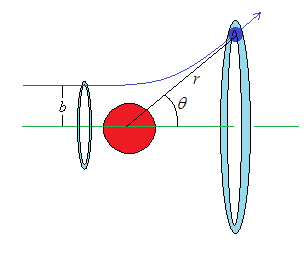
I could try to generalize our definition of P(Ω) from the scattered current through dΩ/scattered current to just current through (Ω, Ω+dΩ)/incident current. Then we’d have:



and this matches previous expression when b = R. But observe that this probability distribution doesn’t really function well as such, since its got a jump discontinuity at a single point, θ = 0. But a single point won’t contribute to an integral, so we can’t really define a probability distribution encompassing *all* angles. Only a p.d.f. encompassing the angles between (0,π] will make any sense – as emphasized above.

**Example: Coulomb Scattering**

We would find that sending a positive charged object initially towards another positively charged object results in the following trajectory.



We worked out the scattering angle previously, but let’s do a quick and dirty calculation again. So let’s go to the orbit equation.



For our type of orbit, r will only exist for φ in the range φ ∈ (π, θ). Now our initial conditions are φ0 = π, r0 = ∞. Filling this in we have:



And so now,



Clearly r(θ) → ∞ again, when:



So that’s our scattering angle. We can simplify a bit,



So,



But let us now work out what ε is. First,



The angular momentum is:



and the constant k is, for the electric force:



So the eccentricity is:



Filling this in we have:



we can solve for b,



And so,



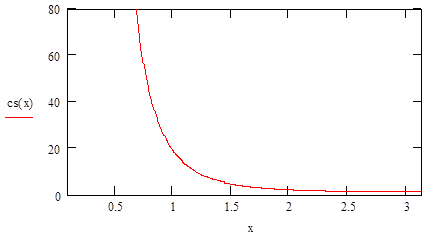
And this yields:



So we have:



Plot of this is:



We can see that forward(er) scattering is overwhelmingly most probable. Having larger T’s will encourage forward(er) scattering. But the weight for reverse scattering is non-zero (it is 1 actually). We can accommodate a finite beam width in our result. Let the radius of the incident cross section be bmax so that σ= πbmax2 (this isn’t actually as obvious as it seems b/c σ = ∫(dσ/dΩ)dΩ, and this can be the cross section area of the target in hard sphere scattering, πR2, or 4πR2 if using QM, among other things? Maybe it is obvious). Then the min scattering angle would be:



And so the probability distribution would be:



Or could’ve said, using the Heaviside step function Θ(x)…



and then,



Just to be on the safe side, let’s check that ∫P(Ω)dΩ = 1.



Nice.

**Example**

Say we send a beam of particles alpha particles into a Gold nucleus with a speed of v = 0.01c. Let the effective cross section radius of the beam be 144pm. What is the probability that particles will forward scatter somewhere within the solid angle θ < 1o?

Well, the probability distribution for the angle is:



where,



The minimum angle comes out to be:



So we want to calculate,



This works out to:

